Math1010C Term1 2016
Tutorial 1, Sept 12

One may wonder why we need to consider sequence. In the first tutorial, we talked about Newton's method and tried to give some reason. Also, we applied Newton's method to approximate $\sqrt{2}$ and check that the sequence does converge to the desired number, hence approximating $\sqrt{2}$.

1 Newton's method

Definition 1. *Zero of a function : Let f be a function. z is said to be a zero of f if $f(z) = 0$*

Given a real-valued function f defined on $\mathbb{R}$(Real number), one may want to approximate the zero of f. Let z denote the zero of f that we want to approximate. Consider the curve $y = f(x)$ on the xy-plane. Choose x_1 to be a number reasonably close to z and draw a tangent line at x_1. Since the tangent line approximates the curve locally, we let the tangent line to pretend the curve. While the value of z may be hard to find, the point that the tangent line cuts the x-axis is easy. We denote that point by x_2.
Iterating the process:
We draw a tangent line at x_2 and denote x_3 the point that the tangent line cuts the x-axis.
We draw a tangent line at x_n and denote x_{n+1} the point that the tangent line cuts the x-axis for n=3,4,5,6....
Eventually, we obtain a sequence $\{x_n\}_{n=1}^{\infty}$
Under some conditions on f, $\lim_{n\to\infty} x_n = z$
Exercise: Write x_{n+1} in terms of x_n, $f(x_n)$ and $f'(x_n)$

1.1 Example

Definition 2. *Monotone decreasing: a sequence $\{x_n\}_{n=1}^{\infty}$ is said to be monotone decreasing if for each natural number n,we have $x_{n+1} \leq x_n$.*

We want to approximate $\sqrt{2}$, so we consider the function $f(x) = x^2 - 2$. In this case, the zeros of f are $\sqrt{2}$ and $-\sqrt{2}$. Using the notation in preceding paragraph, we want to approximate $z = \sqrt{2}$. Drawing the graph of $y = f(x)$ on your own and try to apply the first step of Newton's method for different x_1, then you should be convinced to choose $x_1 > \sqrt{2}$. Applying Newton's method on your graph, you should see that the sequence x_n comes closer and closer to $\sqrt{2}$ and the sequence is monotone decreasing. Let's give the details of the proof.
Some explanation: We need to show that the sequence converges because the preceding description of Newton's method does not guarantee the convergence at all. We just say "under some conditions on f".

Some ingredients we may need:

Theorem 1 (MCT)**.** *If a monotone decreasing sequence has a lower bound, then the sequence converges to a real number.*

Second, for any $r > 0$, $r + \frac{1}{r} \geq 2$. Equality holds if and only if r=1.

Remember that we choose $x_1 > \sqrt{2}$. By Newton's method, x_{n+1} is defined to be $x_{n+1} = -\frac{x_n^2-2}{2x_n} + x_n = \frac{x_n^2+2}{2x_n}$ for n=1,2,3...
First, it is well-defined for each x_{n+1} because $x_n > 0$ for n=1,2,3...
Then, $x_{n+1} - x_n = -\frac{x_n^2-2}{2x_n}$, which is less than or equal to 0 iff $x_n^2 \geq 2$
By the second ingredient, one may conclude that $x_n^2 \geq 2$ for each n=1,2,3...
So, $\{x_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence bounded below by 0
By MCT, $\{x_n\}_{n=1}^{\infty}$ has a limit, which we denote by x.
Since $2x_nx_{n+1} = x_n^2 + 2$ and limit of product equals product of limits, we conclude that $x^2 = 2$. Also, $x_n > 0 \Rightarrow x \geq 0$. Therefore, $x = \sqrt{2}$.

1.2 Follow up

1. We've said that the tangent line approximates the curve locally. Consider $x \neq x_1$, let $h(x)$ be the distance between the curve at x and the tangent . See how $\frac{h(x)}{x-x_1}$ behaves as $x \to x_1$

2. Now, fix $0 < a < 1$ and let $x > 0$ such that $x^2 < a$. We look for $h > 0$ such that $(x+h)^2 < a$. Try to give h in terms of a and x in a formula as simple as possible, for example, both a and x are not

divisors in the formula. let $x_1 = x$ and denote $x_2 = x + h$. Repeat finding new h for new $x = x_2$. Denote $x_3 = x + h$. By iteration, we get an increasing sequence bounded above. Can you conclude that the sequence converges to $\sqrt{a}$?

3. If f is a continuously differentiable function such that $f' > 0$ and f' is increasing, then it seems that Newton's method works. Prove it. Or come back after you learn about mean value theorem.This implies that our sequence in Example1.1 converges. Why?